A NEW ALGORITHM FOR THE COMPUTATION OF GRAVITATIONAL ATTRACTION DUE TO IRREGULARLY SHAPED BODIES

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РЕЗЮМЕ

Тело неправильной формы разделяется на кубы, которые в свою очередь сужываются на точеные массы. Преобразование Фурье в двух переменных для вертикальной компоненты притяжения точечного источника известно в аналитической форме, влияние всех точечных масс, расположенных на одной и той же глубине, может быть задано посредством свёртки. Свёртка в области частот соответствует операции умножения, поэтому Фурье преобразование слоя, состоящего из кубов расположенииых на равных глубинах, легко может быть определено. Эти суммируются с целью получения Фурье преобразования функции, описывающей притяжение от всех точечных масс. Конечный результат получается посредством обратного фурье преобразования.

Скорость вышеупомянутой процедуры примерно на порядок больше чем у известных из литературы других методов. Точность при этом может быть как угодно повышена с увеличением числа точек, заменяющих тело.
Проледура сравнивается с методами Талвани и Евинг (1960) а также и Муфти (1975). Практическое применение проледуры иллюстоируется на двух примерах.

SUMMARY

The irregularly shaped body is divided into cubes and the cubes are concentrated in mass points. The-two-dimensional Fourier transform of the vertical component of the gravitational attraction due to a point source is given analytically, the contribution of all mass points at the same depth can be described by convolution. Convolution corresponds to multiplication in the frequency domain therefore the Fourier transform of each layer of cubes can be easily determined. These are summed to give the Fourier transform of the attraction due to all mass points. The result is obtained by inverse Fourier transformation.

The procedure is faster than the others known from the literature by about one order of magnitude. Accuracy can be made as high as necessary by applying properly dense spacing of points. The procedure is compared to those proposed by Talwani and Ewing (1960) and by Mufti (1975). Practical applications are illustrated by two examples.

Introduction

An algorithm is described for the rapid computation of the gravitational attraction due to irregularly shaped bodies. A number of methods with the same purpose have been proposed (e. g. Talwani and Ewing, 1960., Nagy, 1966, Botezatu et al., 1971, Mufti, 1975). The best of these methods is the latest one, published by Mufti but it is still not easy to implement and could be considerably improved.

The basic sources of improvement are the substitution of the body by a sufficiently large number of mass points and the utilization of the Fast Fourier Transform (in the followings FFT) algorithm. The first is an approximation which is justified when the spacing of the mass points is properly dense.

The starting point of the derivation is common with many previously developed methods known from the literature. The irregularly shaped body is divided into rectangular prisms. Most authors then compute the gravity fields of the prisms. Assuming that the prisms are small compared to the dimensions of the body the sum of their individual gravity fields gives a good approximation to the gravity field of the whole body. The gravity field due to a rectangular prism can be given analytically. Various formulas have been derived, none of them is easy to handle. The repeated evaluation of one of these formulas involves time-consuming computations which practically prohibits its use when the number of prisms are large and the gravity field is to be computed at many points. The gravity field due to a cube, however, can be approximated by the gravity field due to a sphere or a mass point in its center assuming that certain requirements are satisfied. The requirements can always be met by choosing sufficiently small cubes.

The two dimensional Fourier transform of the vertical component of the gravitational attraction due to a point source is also known analytically. Consider now all those points which are at the same depth. The total effect of these points is described by a convolution which includes the masses and their coordinates on the one hand and the effect of a single point of unit mass on the other. The convolution can be computed very fast by using the analytical expression for the effect due to a mass point and the two-dimensional FFT algorithm.

A similar but restricted version have been described by the present author (M e s k ó, 1976) for sheet-like bodies, i. e. for those bodies which are relatively thin and horizontally large. The restrictions have now been removed. The single layer of prisms with various heights, used in the cited paper, have been substituted by the necessary (but arbitririly large) number of cubes.

The difference between the present procedure and Mufti's procedure is twofold. First, always the approximate formula is used for the dimensions of the cubes are chosen according to the requirements. The use of sufficiently small cubes makes superfluous the iterative procedure suggested by M u f t i (1975). The second deviation which results is considerable gian is that the convolution is computed via the FFT.

The procedure has many applications. It can be used e. g. in gravity modelling, in combined interpretation of seismic, gravity and magnetic data and in the solution of the inverse gravity problem.

The algorithm is described in Part I., while some examples including the investigation of the accuracy are given in Part II.

PART I.

The computational algorithm

Before embarking upon the detailed description of the algorithm the approximation of the gravity field due to a cube by the gravity field due to a mass point is discussed.

Let us consider a cube with its center located in the origin of a rectangular coordinate system (z, y, z) and with sides parallel to the coordinate axes. The z-component of the gravitational attraction due to the cube is considered in a point $P(x, y, z)$ outside the body. Let us introduce dimensionless distances by the definitions

$$z' = \frac{z}{a} \quad \text{(vertical distance)}$$

and

$$r'_h = \left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2\right]^{1/2} \quad \text{(horizontal distance)},$$

where 2a denotes the sidelength of the edges of the cube.

Mufti (1973) has shown that the approximation obtained by concentrating the mass of the cube in the origin has an error equal to or less than $\pm 0.1\%$ when

$$z' \geq 3 \quad \text{and} \quad r'_h \geq 6.25 - z' \tag{1a}$$

or

$$z' < 3 \quad \text{but} \quad r'_h \geq 3.25 . \tag{1b}$$

The grid spacing in routine gravity measurements is usually 0.5 km or 1 km. The gravity field of an assumed irregularly shaped body is obtained in a form directly comparable to the measurements when 2a is equal to the grid spacing. Inequality (1a) is obviously satisfied for any horizontal distance if $z' > 6.25$ or $z > 6.25$ a. Then the depth of the upper boundary should be greater than 1.31 km (for 0.5 km spacing) or 2.62 km (for 1 km spacing). Taking into consideration the depths encountered in practical applications we might draw the conclusion that cubes with sidelengths of 0.5 km almost always but in some cases even cubes with sidelengths of 1 km assure proper approximation. In the few remaining cases a finer subdivision can be applied where

$$a \leq \frac{h}{5.25} \tag{2}$$

(h is the depth of the upper boundary).

Let us assume in the followings that the body has a flat base i. e. the lower boundary is a plane at depth H and the upper boundary is described by $U(I,K) \cdot (U(I,K)$ gives uniformly digitized depth data).

The procedure is similar when the upper boundary is a plane (flat top bodies) and the lower boundary is given by the depth data $U(I,K)$ It is also obvious that a body of completely irregular shape can be built by these two type of elements. Therefore the somewhat restricted version of a body with arbitrary upper surface and flat base will be discussed.

A cube is considered to belong to the building elements of the body if its mass center is located somewhere between the upper and lower boundaries. We start at the flat base. The mass centers of the deepest layer of cubes are at the depth

$$H-a\,,$$

the mass centers of the next layer of cubes are at the depth

$$H-3a\,,$$

in general, the mass centers of the n-th layer of cubes are at depth

$$H-(2n+1)\,a\,. \tag{3}$$

The depth of the uppermost layer is determined by the largest $n_{\max}$ satisfying the inequality

$$H-(2n_{\max}+1)\,a \geq \min U(I,K)\,. \tag{4}$$

Assuming a homogeneous density distribution for the body the mass points in the n-th layer can be described by

$$M_n(I,K) = \varrho(2a)^3\,\vartheta_n(I,K)\,, \tag{5}$$

where $\vartheta_n(I,K)$ = 0, if the center of cube is not in tie body,
= 1, if the center of cube is within the body.

The depth of the upper boundary of the body at the point (I,K) is considered the average of the depths in four points of the closest neighbourhood, i.e.

$$\overline{U}(I,K =$$
$$= \frac{1}{4}[U(I+a,K+a)+U(I+a,K-a)+U(I-a,K+a)+U(I-a,K-a)].$$

The value of $\vartheta_n(I,K)$ is determined according to the following inequality

$$\vartheta_n(I,K) = 1\,, \quad \text{for} \quad \overline{U}(I,K) < H-(2n+1)\,a\,, \tag{6a}$$

$$= 0\,, \quad \text{for} \quad \overline{U}(I,K) \geq H-(2n+1)\,a\,. \tag{6b}$$

Other, more sophisticated criteria could have been used to decide whether a cube belongs to the body or not but the simple criteria (6a), (6b) proved to be satisfactory.

The vertical component of the gravity field at the point (J,L) of the surface due to a point mass $M_n(I,K)$ in the n-th layer is given by

$$g_{n,I,K}(J,L) = \frac{G\,\varrho(2a)^3 H'}{\{(2a)^2[J-I)^2+(L-K)^2]+(H')^2\}^{3/2}}$$

where $H' = H-(2n+1)a$. If ϱ is given in gcm^{-3}, the distances in kms, and g is to be obtained in milligals

$$g_{n,I,K}(J,L) = 6.67\,\varrho(2a)^3 \frac{H'}{\{(2a)^2[(J-I)^2+(L-K^2)]+(H')^2\}^{3/2}}. \quad (7)$$

The contribution of the n-th layer to the vertical component of the gravity field is obviously given by the convolution

$$g_n(J,L) = \sum_I \sum_k g_{n,I,K} = M_n(I,K) * S_n(I,K)\,, \quad (8)$$

where

$$S_n(I,K) = \frac{H'}{\{(2a)^2[I^2+K^2]+(H')^2\}^{3/2}}. \quad (9)$$

The Fourier transform of (9) is

$$F\{S_n(I,K)\} = e^{-2\pi H'(fx^2+fy^2)^{1/2}}\,,$$

therefore the Fourier transform of the contribution from the n-th layer by the convolution theorem yields as

$$6.67 \cdot F\{M_n(I,K)\}\, e^{-2\pi H'(fx^2+fy^2)^{1/2}}\,, \quad (10)$$

(where f_x and f_j denote spatial frequencies).

The Fourier transform of the sum of the contributions due to all layers is the sum of the Fourier transforms given by (10). The vertical component of the gravitational attraction is obtained at last by taking the inverse Fourier transform of the sum as follows

$$g(J,L) = 6.67\,F^{-1}\left\{\sum_{n=0}^{n_{max}} F\{M_n(I,K)\}\, e^{-2\pi H'(fx^2+fy^2)^{1/2}}\right\}. \quad (11)$$

The two transformations can be computed very fast by the FFT, the determination of the $M_n(I,K)$ by (6a) and (6b) and the computation of the exponential expression in (11) do not require much computer time. As a result the whole procedure is much faster than those known from the literature. One order or two orders of magnitudes are fair estimates (depending on the dimensions of the body).

PART II.

Investigation of the accuracy, practical examples

The suggested procedure has been compared to those described in the literature. Because Mufti's method is the latest and best method known to the present author this has been chosen as a reference.

M u f t i (1975) computed the gravitational attraction due to a sphere with a radius of 3000 feet and with its center at 4000 feet for a density 0.25 gcm^{-3}. The vertical component of the gravitational attraction has been determined as the sum of the lower and upper halves. Results obtained by the present method are shown in Figures 1. and 2. The gravity field due to a homogeneous sphere can be given analytically as

$$g(r_h) = MG\frac{H}{(r_h^2+H^2)^{3/2}}.$$

Deviations between the numerically computed values and those obtained from the evaluation of the simple analytical expression can be used to check the accuracy of various numerical methods.

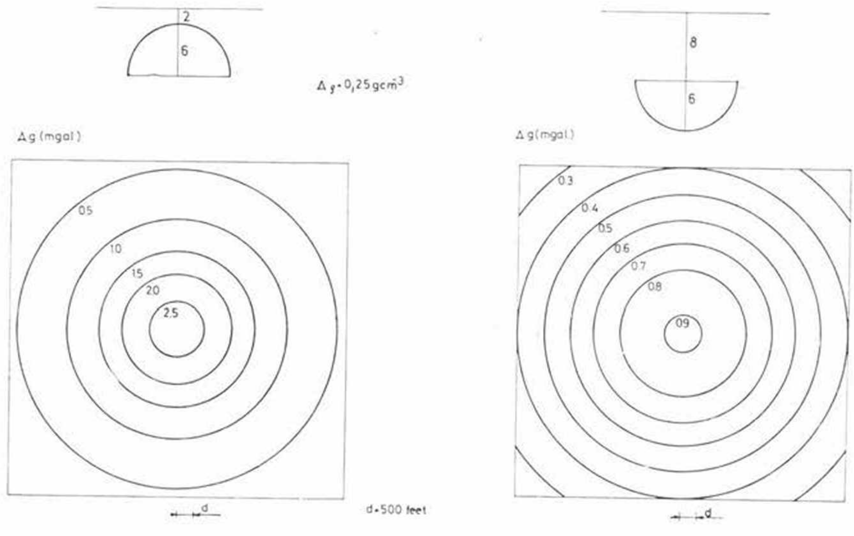

Fig. 1. Vertical component of the gravitational attraction due to half spheres. Distances indicated on the upper models are given in 500 feet units

Figure 3. shows the deviations for 4 methods. The cross section where analytical and numerical values are compared goes through the maximum of the anomaly. Both in the Talwani-Ewing's method and in the author's method 24 slices or 24 layers of cubes have been assumed to obtain a better comparison. It may be mentioned, however, that a subdivision with 48 layers of cubes have also been computed by the present method for the whole grid (32×32 points) in less time that the T a l w a n i - E w i n g method requires for a single point. Figure 3. clearly shows that the author's method is generally more accurate even in the pictured version (24 layers) than the others. There are two exceptions, the origin and the point at the distance of 1000 feet from the origin.

Errors of a 48 layer subdivision on the other hand, are always less than 1 μgal therefore could not be pictured in Figure 3.

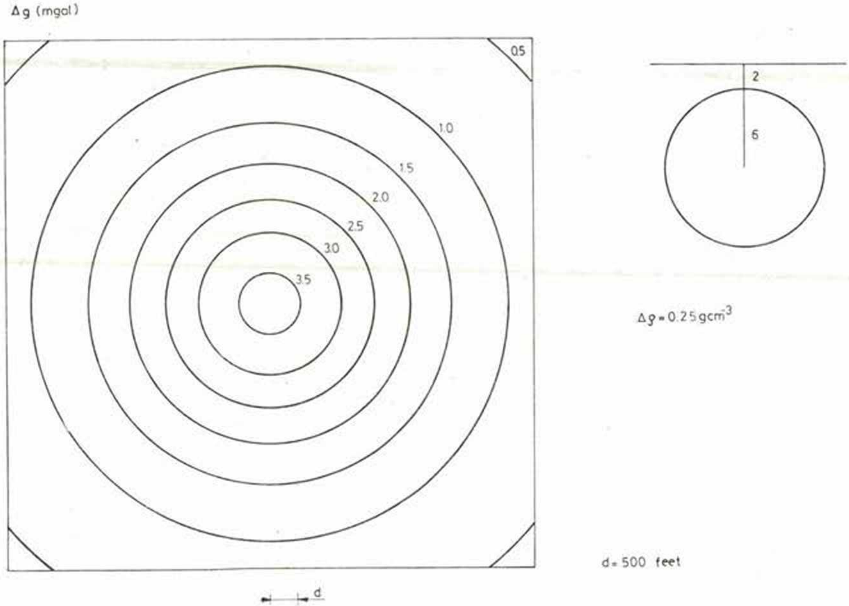

Fig. 2. Vertical component of the gravitational attraction due to a sphere as computed by the present algorithm. Distances indicated on the model are given in 500 feet units

Table 1. summarizes the numerical values.

Table 1.

Comparison of the vertical component of the gravity field computed by various methods
Model: sphere, radius: 1000 feet, depth of center: 3000 feet, density 0.25 cgs unit
Section: through the center of the anomaly

Distance from the origin measured along the x-axis in kilofeet	Analytical results (in mgals)	Talwani–Ewing method		Mufti, (1975) method		Meskó method	
		result (mgal)	error (μgal)	result mgal	error (μgal)	result mgal	error μgal
0	3.5926	3.6023	9.7	3.5947	2.0	3.6061	+13.5
1	3.2803	3.3099	29.6	3.2761	− 4,2	3.2845	+ 4.2
2	2.5707	2.6066	35.9	2.5530	−17.7	2.5696	− 1.1
3	1.8394	1.8702	30.8	1.8272	−12.2	1.8375	− 1.9
4	1.2702	1.2940	23.8	1.2627	− 7.5	1.2676	− 2.6
5	0.8758	0.8929	17.1	0.8710	− 4.8	0.8734	− 2.4
6	0.6132	1.6260	12.8	0.6098	− 3.4	0.6113	− 1.9
7	0.4388	0.4477	8.9	0.4363	− 2.5	0.4374	− 1.4
8	0.3213	0.3279	6.6	0.3196	− 1.7	0.3204	− 0.9
9	0.2407	0.2456	4.9	0.2395	− 1.4	0.2400	− 0.7
10	0.1840	0.1877	3.7	0.1830	− 1.0	0.1835	− 0.5

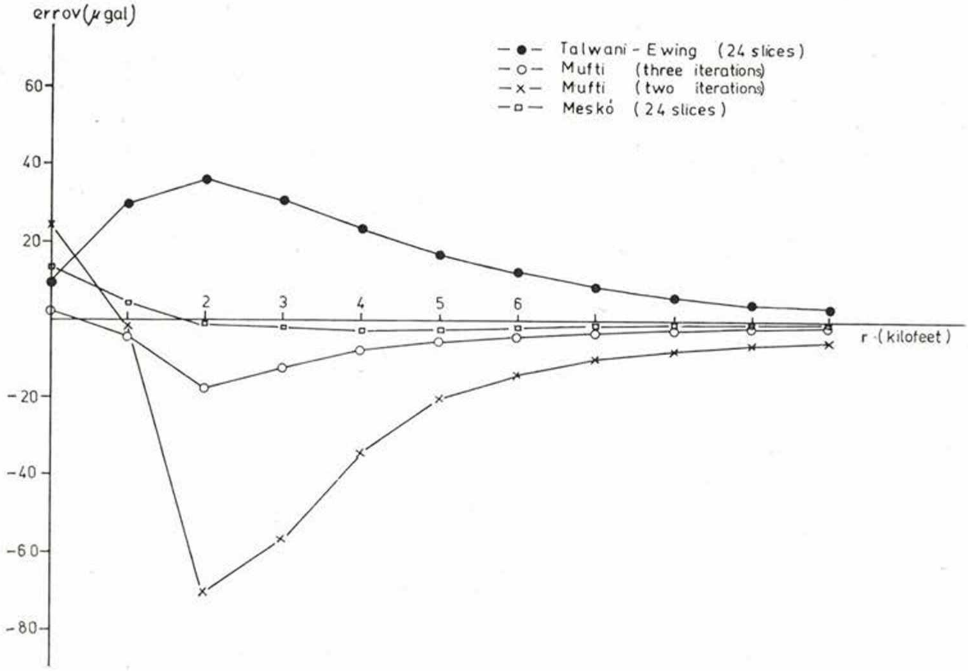

Fig. 3. Comparison the accuracy of various methods. Deviations between the analytical formula and various methods are plotted against the distance from the top of the anomaly

Figure 4. shows a model structure and its gravity field. The flat base is at the depth of 2 kms. The contours of the structure (height of the upper boundary above the flat base) are shown in the left hand side

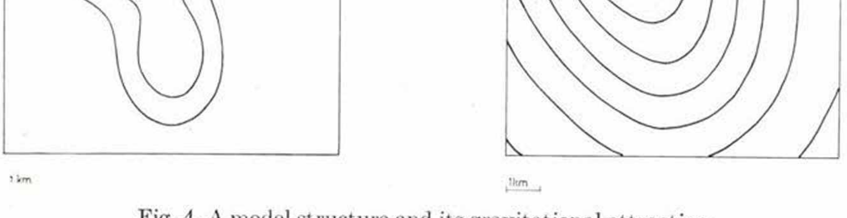

Fig. 4. A model structure and its gravitational attraction

of Figure 4. The density is 0.2 gcm^{-3}, which corresponds to the density contrast between a realistic volcanic body and its sedimentary surroundings.

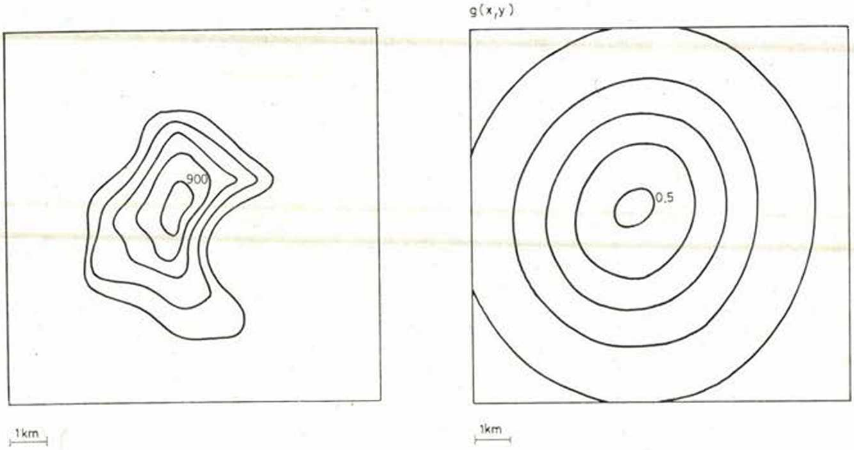

Fig. 5. A real geological structure determined by seismic reflections and its gravity field

Figure 5. shows a real geological structure determined by the reflection seismic method. The flat base is now at 3100 ms and the contours of the upper boundary (100, 300, 500, 700 and 900 meters above the plane) are given on the left hand side. A density contrast of 0.1 gcm^{-3} has been assumed. The computed gravity field (in mgal units) is shown on the right hand side.

The speed of the proposed method makes possible the experimentation with a large number of geological models. It is the hope of the author that the easy computability of gravity anomalies due to various models will influence advantageously the gravity interpretation as well as the combined interpretation of gravity, magnetic and seismic data.

REFERENCES

Botezatu et al. (1971): Approximation of the gravitational attraction of geological bodies Geophys. Prosp. Vol 19., pp. 218 – 227.

Meskó A. (1976): An iterative solution of the inverse gravity problem for constrained models Annales, Vol XIX., pp. 83 – 113.

Mufti, I. R. (1973): Rapid determination of the cube's gravity field. Geophys. Prosp. Vol 21., pp. 724 – 736.

Mufti, I. R. (1975): Iterative gravity modeling by using cubical blocks Geophys. Prosp. Vol 23., pp. 163 – 199.

Nagy, D. (1966): The gravitational attraction of a right rectangular prism Geophysics, Vol 31., pp. 362 – 371.

Talwani, M. and Ewing, M. (1960): Rapid computation of gravitational attraction of three – dimensional bodies of arbitrary shape Geophysics, Vol 25., pp. 203 – 225.